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The Thermodynamic Bethe Ansatz and the 1/N Correction to the Density Phase Transition in the Gross-Neveu Model

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Abstract

We compute the $1/N$ correction to the location of the previously found first-order phase transition in the Gross-Neveu model at a chemical potential $\mu = \mu_c = \frac{1}{\sqrt{2}}m$, where m is the fermion mass. We employ an expression for the free energy $f(\mu)$ given by the thermodynamic Bethe ansatz under the approximation that the fundamental fermions dominate the ground state, and combine it with the effective potential evaluated at zero chemical potential. Our result is $\mu_c = \frac{m}{\sqrt{2}}[1 - \frac{0.47}{N}]$.

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1. Introduction

In a previous paper [1], we studied the Gross-Neveu model [2] in the presence of a chemical potential μ. We computed the effective potential analytically to leading order in $1/N$, and we found a first-order phase transition at

$$\mu = \mu_c = \frac{1}{\sqrt{2}} m. \tag{1}$$

where m is the fermion mass. For $\mu > \mu_c$ the discrete chiral symmetry is restored and the fermion becomes massless. The earlier references on the phase transition of the Gross-Neveu model may be found in ref. [3].

It is known [4] [5] that the $N = 2$ Gross-Neveu model provides an approximate description of trans-polyacetylene [6], and we used our results to explain an observed phase transition in polyacetylene as a function of doping concentration (which is related to the chemical potential) [7]. We found agreement with the order, the nature, and the location of the phase transition, and we concluded that the Gross-Neveu model did indeed capture some of the essential physics of trans-polyacetylene.

One aspect of this work that requires further amplification, however, is the fact that polyacetylene corresponds to the $N = 2$ Gross-Neveu model, whereas our results were derived only to leading order in $1/N$. This raises the possibility that the next leading term in the $1/N$ expansion might provide a quantitatively significant correction, which would then render the excellent agreement between theory and experiment found in leading order merely fortuitous.

In this paper we address the question of the order $1/N$ correction to our previous computation. The straightforward way to do this would be to evaluate the Feynman graphs that contribute to order $1/N$, incorporating, as we did in leading order, the effect of μ into a modification of the fermion propagator. However, the evaluation of the $1/N$ correction to the effective potential even when $\mu = 0$ is a rather involved affair, and the extension to non-zero μ appears to be prohibitively complicated. Instead, we shall evolve a technique based on the thermodynamic Bethe ansatz (TBA) [8] [9].

In the next section, we shall explain this technique and illustrate it by rederiving the results of our previous paper. Then in sections 3 and 4, we shall extend the computation to next order in $1/N$; for simplicity, we shall find it convenient to make a further approximation in which we restrict the Gross-Neveu S-matrix to a single component. This will be explained more fully below. In section 5 we shall summarize our results and draw some conclusions.

2. The TBA in Leading Order

The TBA has been much used recently to discuss the properties of exactly integrable models [9], [10]. In this paper, somewhat unconventionally, we shall actually use it to determine the thermodynamic properties of a particular system. The TBA should be very useful technique in discussing thermodynamics of the Gross-Neveu model because all the S-matrix elements are known [11] [12] [13].

The basic quantity of interest is the "dressed" excitation energy $\tilde{\epsilon}(\theta)$ that is required to promote a single particle from the Dirac sea to an energy $mcosh\theta$; $\tilde{\epsilon}(\theta)$ includes the effects of the rearrangement of the vacuum after the particle has been excited. At finite temperature, the equation for $\tilde{\epsilon}(\theta)$ is non-linear, but at zero temperature (which is the case of interest to us) it reduces to the linear equation [14]

$$\tilde{\epsilon}(\theta) = \mu - mcosh\theta + \int_{-B}^{B} d\theta' K(\theta - \theta')\tilde{\epsilon}(\theta'). \tag{2}$$

Here the kernel K is the logarithmic derivative of the S-matrix:

$$K(\theta - \theta') = (\frac{1}{2\pi i})\frac{d\ lnS(\theta - \theta')}{d\theta} \tag{3}$$

and B is determined by the condition

$$\tilde{\epsilon}(\pm B) = 0. \tag{4}$$

Effectively this condition determines B as a function of the chemical potential μ.

Once $\tilde{\epsilon}$ is known, the free energy f of the system can be determined as a function of μ up to an additive constant:

$$f(\mu) - f(0) = -(\frac{m}{2\pi}) \int_{-B}^{B} d\theta \tilde{\epsilon}(\theta) cosh\theta. \tag{5}$$

As shown in ref.[11], the expansion of K in powers of $1/N$ begins in order $1/N$, so to leading order one ignores the integral on the right-hand side of eq. (2), and one has immediately

$$\tilde{\epsilon}(\theta) = \mu - mcosh\theta \tag{6}$$

and hence

$$coshB = \mu/m. \tag{7}$$

This is possible only if $\mu \geq m$; if $\mu < m$, no solution for $\tilde{\epsilon}$ exists.

¿From eq. (5) we find

$$\begin{aligned} f(\mu) - f(0) =& \theta(\mu^2 - m^2)(\frac{m^2}{2\pi})(B - \frac{1}{2}sinh2B) \\ =& \theta(\mu^2 - m^2)(\frac{m^2}{2\pi})[ln(\mu/m + \sqrt{\mu^2/m^2 - 1}) - \mu/m\sqrt{\mu^2/m^2 - 1}]. \end{aligned} \tag{8}$$

On dimensional grounds, $f(0) = -bm^2$ where b is a dimensionless constant, i.e.

$$f(\mu) = -bm^2 + \theta(\mu^2 - m^2)\frac{m^2}{2\pi}[ln(\mu/m + \sqrt{\mu^2/m^2 - 1}) - \mu/m\sqrt{\mu^2/m^2 - 1}]. \tag{9}$$

At first sight, there appears to be no way that this expression can be used to provide evidence for a phase transition at $\mu = m/\sqrt{2}$, since $f(\mu)$ is absolutely flat for $\mu < m$. We observe, however, that if a massless phase exists, then its free energy $f_0(\mu)$ is given by eq. (9) with m=0:

$$f_0(\mu) = -\mu^2/2\pi. \tag{10}$$

Provided that $0 < b < 1/2\pi$, the curves $f(\mu)$ and $f_0(\mu)$ will intersect at

$$\mu^2/m^2 = 2\pi b, \tag{11}$$

with $f_0(\mu) > f(\mu)$ for $(\mu/m) < 2\pi b$. One therefore predicts a first-order phase transition from the massive phase to the massless one at $\mu/m = \sqrt{2\pi b}$.

The one piece of information the TBA does not give us is the value of b. For this, we need the effective potential computation, but only at $\mu = 0$. Thus the TBA plus the effective potential at $\mu = 0$ allows us to determine the phase structure of the theory for $\mu > 0$.

To make contact with previous work, we note that the free energy as given above corresponds to the value of the effective potential V at its minimum σ_{min}, and the mass m is equal to $g\sigma_{min}$ where g is the GN coupling constant. We can verify this explicitly by appealing to eqn. (20) of ref. [1], from which we obtain

$$V_{eff}(\sigma) = \frac{\sigma^2}{2N} + \frac{\sigma^2}{4N}[\theta(\sigma^2-\gamma^2)(ln(\sigma^2/\sigma_0^2)-3) + \theta(\gamma^2-\sigma^2)(2ln\frac{\gamma+\sqrt{\gamma^2-\sigma^2}}{\sigma_0}-3)] \\ - \frac{\gamma}{2N}\sqrt{\gamma^2-\sigma^2}\theta(\gamma^2-\sigma^2)\ . \tag{12}$$

This expression comes equipped with the following comments:

(i) We have divided V_{eff} of ref. [1] by N. This ensures a finite limit as $N \to \infty$ (with g^2N and m fixed) and also is the correct normalization to correspond to $f(\mu)$, as we shall see explicitly below;

(ii) We have omitted the last term of eq. (20) of ref. [1], since it is precisely cancelled by another contribution that was left out of eq. (20), called $\tilde{V}$ in eq. (14) of ref. [1]. Thus V_{eff} above includes all the relevant contributions;

(iii) The parameter γ above is $\mu\sqrt{N/\pi}$.

(iv) We have set the coupling $\lambda = g^2N$ that appears in eq. (20) of ref. [1]equal to π. This is permissible because physics depends only on

$$\sigma_{min} = \sigma_0 e^{(1-\pi/\lambda)} \tag{13}$$

where σ_0 is an arbitrary renormalization point. The choice $\lambda = \pi$ is convenient because then $\sigma_{min} = \sigma_0$. We see that

$$V_{eff}(\sigma_0) = -\frac{\sigma_0^2}{4N} + \theta(\gamma^2-\sigma_0^2)[\frac{\sigma_0^2}{2N}ln(\frac{\gamma+\sqrt{\gamma^2-\sigma_0^2}}{\sigma_0}) - \frac{\gamma\sqrt{\gamma^2-\sigma_0^2}}{2N}]\ . \tag{14}$$

Putting $m^2 = g^2\sigma_0^2 = \frac{\pi\sigma_0^2}{N}$, and $\mu^2 = \frac{\pi\gamma^2}{N}$ we have

$$V_{eff} = -\frac{m^2}{4\pi} + \theta(\mu^2-m^2)\frac{m^2}{2\pi}[ln\frac{\mu+\sqrt{\mu^2-m^2}}{m} - \frac{\mu}{m}\sqrt{\frac{\mu^2}{m^2}-1}] \tag{15}$$

which tallies exactly with the expression for $f(\mu)$, as given by the TBA, provided $b = 1/4\pi$ (and hence $\mu_c = m/\sqrt{2}$).

Of course, as pointed out above, we did not need the full V_{eff}, including the effects of μ, to determine b. It would have sufficed to know the original $V_{eff}(\sigma)$, computed by Gross and Neveu [2]:

$$V_{GN}(\sigma) = \frac{\sigma^2}{2N} + \frac{\sigma^2}{4N}[ln(\sigma^2/\sigma_0^2) - 3] \tag{16}$$

from which $V_{GN}(\sigma = \sigma_0 = \sqrt{\frac{N}{\pi}}m) = -\frac{m^2}{4\pi}$ follows immediately as well.

3. Next to Leading Order

In next order, we of course do not have the explicit form of V_{eff} as a function of σ and μ. For $\mu = 0$, the $1/N$ correction was computed long ago by Schonfeld [15] and by Root [16]. Their results will allow us to extract the $1/N$ correction to the value $b = +\frac{1}{4\pi}$ found in the previous section. The other ingredient we shall need is the form of $f(\mu)$ obtained from the TBA, especially the massless limit $f_0(\mu)$. We shall consider each of these in turn.

For our purposes, it is convenient to follow the formulation of the $1/N$ correction ΔV to the Gross-Neveu model given by Schonfeld. By summing Feynman diagrams, he obtains an unrenormalized expression.

$$\Delta V^{un} = \int_0^\infty dx \left\{ ln[\frac{1}{2}A + ln(\sigma/\sigma_0)] - ln\ ln(\frac{x}{g^2\sigma_0^2}) \right\} \tag{17}$$

where we have set $\lambda = \pi$ in Schonfeld's expression, and where $A = Z\ ln(\frac{Z+1}{Z-1})$, and $Z = [1 + \frac{4g^2\sigma^2}{x}]^{1/2}$. ΔV^{un} is divergent, and it also has an imaginary part. As Schonfeld explains, the correct physical prescription, at least in the range of σ we are interested in ($\sigma \simeq \sigma_0$) is to take the real part. The divergences are removed by a renormalization prescription which amounts to the conditions

$$Re\Delta V\ |_{\sigma=\sigma_0} = \frac{\partial^2}{\partial\sigma^2}(Re\Delta V)\ |_{\sigma=\sigma_0} = 0\ . \tag{18}$$

Now in leading order, $\sigma_{min} = \sigma_0$ (since $\lambda = \pi$), and so we must have

$$\sigma_{min} = \sigma_0 + \frac{1}{N}\sigma_1, \quad \text{with} \quad V = V_{GN} + Re\Delta V \tag{19}$$

and

$$V(\sigma_{min}) = V_{GN}(\sigma_0 + \frac{1}{N}\sigma_1) + Re\Delta V(\sigma_0 + \frac{1}{N}\sigma_1)\ . \tag{20}$$

But $V'_{GN}(\sigma_0) = 0$, and ΔV is already order $1/N$, and so to the order we are working,

$$\begin{aligned} V(\sigma_{min}) =& V_{GN}(\sigma_0) + Re\Delta V(\sigma_0) \\ =& V_{GN}(\sigma_0) \qquad\qquad \text{because of eq. (18)} \\ =& -\frac{\sigma_0^2}{4N} \\ =& -\frac{1}{4N}(\sigma_{min} - \frac{1}{N}\sigma_1)^2 = -\frac{1}{4N}(\sigma_{min}^2 - \frac{2}{N}\sigma_{min}\sigma_1)\ . \end{aligned} \tag{21}$$

The quantity σ_1 is determined from the condition

$$V'(\sigma_0 + \frac{1}{N}\sigma_1) = 0 \tag{22}$$

and is given by

$$\frac{\sigma_1}{N} = -Re\Delta V'(\sigma_0) \tag{23}$$

where use has been made of the renormalization condition $V''(\sigma_0) = 1$.

It can easily be seen from Schonfeld's paper that in terms of the unrenormalized expression ΔV^{un} defined above,

$$Re\Delta V'(\sigma_0) = Re\Delta V'^{un}(\sigma_0) - \frac{4}{3\sigma_0}Re\Delta V^{un}(\sigma_0) - \frac{1}{3}\sigma_0\Delta V''^{un}(\sigma_0) \tag{24}$$

where the last two terms on the rhs are the counterterms necessary to implement the conditions (18) and thereby render the expression finite.

The remaining task is the purely numerical one of using the integral expression (17) to evaluate the right-hand side of eq. (24). One finds

$$\frac{\sigma_1}{N} = (\frac{1.06}{3N})\sigma_0 \tag{25}$$

and hence

$$V(\sigma_{min}) = -\frac{1}{4N}\sigma_{min}^2 + \frac{1}{2N^2}(\frac{1.06}{3})\sigma_{min}^2 \tag{26}$$

where in the second term we have used $\sigma_{min} = \sigma_0 + \vartheta(1/N)$.

Hence $V_{min} = -bm^2$ where $b = \frac{1}{4\pi}[1 - \frac{2.12}{3N}]$.

4. The TBA in Next to Leading Order

Let us formulate the TBA in the framework of $1/N$ expansion. While we actually deal only with the next to leading order in this paper, we present the formulas in an organized way so that the structure of the higher orders in $1/N$ becomes transparent. We expand various quantities introduced in section 2 in powers of $1/N$:

$$\begin{aligned} \tilde{\epsilon}(\theta) &= \tilde{\epsilon}_0(\theta) + \frac{1}{N}\tilde{\epsilon}_1(\theta) + \cdots \\ f(\mu) - f(0) &= g_0(\mu) + \frac{1}{N}g_1(\mu) + \cdots \\ K(\theta) &= \frac{1}{N}K_1(\theta) + \frac{1}{N^2}K_2(\theta) + \cdots \end{aligned} \tag{27}$$

One might think that the boundary of the integral B in (2) determined by (4) receives a $1/N$ correction and that it is therefore incumbent on us to expand B in powers of $1/N$. However, by examining a model in which the TBA equation can be solved exactly, we learn that it is more appropriate to keep B fixed and change μ as N is varied. We will treat this in the Appendix. Thus we expand the chemical potential

$$\mu(B) = \mu_0(B) + \frac{1}{N}\mu_1(B) + \cdots \quad . \tag{28}$$

μ_0 is determined by the leading order equation

$$\tilde{\epsilon}_0(B) = 0 \quad . \tag{29}$$

For clarity we recapitulate the leading order result obtained in section 2 in the present notation:

$$\begin{aligned} \tilde{\epsilon}_0 &= \mu_0 - m cosh\theta \\ g_0(\mu) &= \theta(\mu_0^2 - m^2)\frac{m^2}{2\pi}(B - \frac{1}{2}sinh2B) \quad . \end{aligned} \tag{30}$$

Following Forgacs, Niedermayer and Weisz [14], we make the ansatz that the ground state of the Gross-Neveu model at finite density and zero temperature is dominated at large-N by the fundamental fermions. This is a reasonable assumption because the solitons and their bound states are more massive than the fundamental fermions at large N. This ansatz allows us to ignore all the S-matrix elements but the symmetric combination of the fundamental fermion scattering amplitudes, as shown in [14]. The approximation is in fact

implicit in (2); without the ansatz K becomes a matrix which acts on a vector $\tilde{\epsilon}(\theta)$ in the particle species basis.

The consistency of the ansatz is checked by Forgacs, et al., in refs. [14] and [17]. The perfect matching of the large-N and the TBA results in the leading order, as discussed in the preceding section, lends further support to this viewpoint.

With this ansatz, the kernel K in the TBA equation (3) in leading non-vanishing order has the form

$$K_1(\theta) = -\frac{d}{d\theta}(\frac{1}{\theta} - \frac{1}{sinh\theta}). \tag{31}$$

Using (2) it is straightforward to compute the $1/N$ correction of the dressed particle energy $\tilde{\epsilon}(\theta)$ and the free energy $f(\mu) - f(0)$. They read

$$\begin{aligned}
\tilde{\epsilon}_1 =& \mu_1 + msinh\ \theta[Chi(\theta+B) - Chi(\theta-B) - ln\left\{\frac{sinh(\theta+B)}{sinh(\theta-B)}\right\}] \\
& - mcosh\ \theta[Shi(\theta+B) - Shi(\theta-B) - 2B] \\
g_1(\mu) =& -\theta(\mu_0^2 - m^2)\frac{m^2}{\pi}\ [B^2 + sinh^2 B - BShi(2B)]
\end{aligned} \tag{32}$$

where $Shi(x)$ and $Chi(x)$ are the hyperbolic integral functions defined by

$$\begin{aligned}
Shi(x) &\equiv \int_0^x dt \frac{sinh\ t}{t} \\
Chi(x) &\equiv \gamma + lnx + \int_0^x dt \frac{cosh\ t - 1}{t}\ .
\end{aligned} \tag{33}$$

Here γ is Euler's constant. The result for $g_1(\mu)$ in (32) reproduces that of Forgacs, et al. [17].

The boundary condition $\tilde{\epsilon}(B) = 0$ with $\tilde{\epsilon}_0(B) = 0$, which is already met in the leading order, determines μ_0 and μ_1:

$$\begin{aligned}
\mu_0 &= mcoshB \\
\mu_1 &= mcoshB\ [Shi(2B) - 2B\ ] - msinhB\ [Chi(2B) - ln\,\{sinh2B\} - \gamma]
\end{aligned} \tag{34}$$

To implement the strategy explained in section 2 we need the expression for the free energy in the massless limit, which corresponds to $B \to \infty$. Using the asymptotic form of the hyperbolic integral functions

$$
\begin{aligned}
Shi(x) &\rightarrow_{x\rightarrow\infty} \frac{e^x}{2x} \\
Chi(x) &\rightarrow_{x\rightarrow\infty} \frac{e^x}{2x}
\end{aligned}
\tag{35}
$$

one can easily show that

$$
\begin{aligned}
&\lim_{m\rightarrow 0} g_1(\mu) = 0 \\
&\lim_{m\rightarrow 0} \mu_1 = \mu_0(\gamma - ln2) \quad .
\end{aligned}
\tag{36}
$$

The latter implies that

$$
\mu = \mu_0(1 + \frac{\gamma - ln2}{N}) \quad . \tag{37}
$$

Therefore, we obtain

$$
\begin{aligned}
\lim_{m\rightarrow 0} [f(\mu) - f(0)] &= -\frac{\mu_0^2}{2\pi} \\
&= -\frac{\mu^2}{2\pi}[1 + \frac{2}{N}(ln2 - \gamma] \\
&= -\frac{\mu^2}{2\pi}(1 + \frac{0.232}{N})
\end{aligned}
\tag{38}
$$

We deduce by combining (26) and (38), that the phase transition takes place at

$$
\frac{\mu^2}{2\pi}(1 + \frac{0.232}{N}) = \frac{m^2}{4\pi}(1 - \frac{2.12}{3N}) \tag{39}
$$

or

$$
\frac{\mu^2}{m^2} = \frac{1}{2}(1 - \frac{0.94}{N}) \tag{40}
$$

If we estimate (40) at $N = 2$ we have

$$
\mu = \frac{m}{\sqrt{2}}(1 - 0.23) \tag{41}
$$

which implies about a 20% correction to the critical chemical potential due to the next to leading order in $1/N$.

5. Conclusions

In our previous paper, in leading order in the $1/N$ expansion, we found a first-order phase transition in the Gross-Neveu model as a function of chemical potential, occurring at $\mu_c = \frac{1}{\sqrt{2}}m$.

In this paper, making use of the $1/N$ corrections to the $\mu = 0$ effective potential, and also of the thermodynamic Bethe ansatz, we have shown that the above result receives a correction of the form

$$\mu_c = \frac{1}{\sqrt{2}}m[1 - \frac{(.47)}{N}] \quad . \tag{42}$$

In view of the result, it is reassuring to note that the correction we have found acts to lower the value of μ_c, i.e. to make it easier for the system to undergo the phase transition. The validity of our picture might begin to run into trouble if μ_c were to become larger and were to approach m in magnitude, because for $\mu > m$ $f(\mu)$ decreases rapidly from its constant value of $f(0)$, and the question of whether $f(\mu)$ and $f_0(\mu)$ intersect at all would then become more problematical.

When we evaluate our correction for $N = 2$ we find a 20% effect. Since the value of the doping concentration y is, in this model, directly proportional to μ, this implies that y changes from .064 to perhaps .050, which is in somewhat less good agreement with published results, but still well within the experimental limits.

In obtaining the above result, we have made one further approximation: following Forgacs, et al., we have restricted the S-matrix appearing in the TBA equation to a single component, the scattering between fundamental fermions. We believe that it is a good approximation at large-N because all other excitations but fundamental fermions have masses of the order of N. In fact, the result by Forgacs et al. obtained under the same approximation indicates a perfect matching between the TBA and the perturbative calculations even at large values of chemical potential. Nonetheless, we should mention that the validity of the approximation has to be checked by more elaborate computation which includes all the solitons and bound states. The obstacle at present to such computation is a considerable amount of complexity and labor.

Also as we noted above, we have chosen what may seem a curious strategy in performing the $1/N$ expansion for the TBA. Rather than keeping the physical parameter μ fixed, and developing the auxiliary variable B in a series in $1/N$, we have inverted this procedure. Of course, the two must ultimately be equivalent, but we have found, in studying a simple

exactly solvable example of the TBA equation, that the latter is more transparent and more manageable. We amplify this point in a brief appendix.

Another aspect we have discussed in this paper is how the density phase transition between massive and massless phases can be addressed within the framework of the TBA. The TBA itself only allows us to evaluate the difference between free energies in finite and zero chemical potential theories. We have proposed a new machinery of combining the massless limit of the TBA result with that of the computation of the effective potential at zero chemical potential. We have verified that it works in the leading order in the 1/N expansion of the Gross-Neveu model. It remains to be seen if it has wider applicability. It would also be a challenging task to see whether one can develop a formalism that could reveal this type of phase transition solely within the framework of the TBA.

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Appendix A.

We give a brief description of an exactly solvable model of the TBA. The model is defined by the 2-body S-matrix

$$S_{ab} = exp(\frac{2\pi i\lambda}{m^2}\epsilon_{\mu\nu}p_a^\mu p_b^\nu), \qquad (A1)$$

where λ is a real constant. The S-matrix satisfies unitarity and "crossing" but does not have polynomial boundedness, and hence may not be realized in local quantum field theories. We shall utilize the model as a theoretical laboratory to examine the large-N expansion

of the TBA. We nevertheless keep in mind the possibility that it may allow a physical interpretation. Properties of models without polynomial boundedness have recently been discussed by Khuri [18]. With the definition $p_i^\mu = m(cosh\theta_i, sinh\theta_i)$, $i = a, b$, and the introduction of the relative rapidity $\theta = \theta_a - \theta_b$, the kernel K takes the simple form

$$K(\theta) = \lambda cosh\theta. \tag{A2}$$

The TBA equation (2) can be readily solved as

$$\tilde{\epsilon}(\theta) = \mu - \tilde{m}cosh\theta \tag{A3}$$

where

$$\tilde{m} = \frac{m}{1 + \lambda[\frac{1}{2}sinh2B - B]} \tag{A4}$$

Notice that B is determined by (4), $\mu - \tilde{m}coshB = 0$, and we can obtain B nonperturbatively in this model. The free energy of the model can be easily computed as

$$f(\mu) - f(0) = -\theta(\mu - \tilde{m})(\frac{m^2}{2\pi})\frac{1}{\lambda + [\frac{1}{2}sinh2B - B]^{-1}}. \tag{A5}$$

We treat the model perturbatively to gain insight for the large N expansion in the TBA. We expand various quantities in powers of λ:

$$\begin{aligned} \mu &= \mu_0 + \lambda\mu_1 + \lambda^2\mu_2 + \cdots. \\ \tilde{\epsilon}(\theta) &= \tilde{\epsilon}_0(\theta) + \lambda\tilde{\epsilon}_1(\theta) + \lambda^2\tilde{\epsilon}_2(\theta) + \cdots. \\ f(\mu) - f(0) &= g_0(\mu) + \lambda g_1(\mu) + \lambda^2 g_2(\mu) + \cdots. \end{aligned} \tag{A6}$$

But we keep B fixed as in our treatment of the Gross-Neveu model in section 4. Then one can solve the TBA equation order by order in perturbation theory to obtain the results:

$$\begin{aligned} \mu_n &= \mu_0[B - \frac{1}{2}sinh2B]^n, \\ \tilde{\epsilon}_n(\theta) &= \tilde{\epsilon}_0(\theta)[B - \frac{1}{2}sinh2B]^n, \end{aligned} \tag{A7}$$

where

$$\begin{aligned} \mu_0 &= mcoshB, \\ \tilde{\epsilon}_0(\theta) &= \mu_0 - mcosh\theta = m(coshB - cosh\theta). \end{aligned} \tag{A8}$$

It can be readily seen that they reproduce the exact solution (A3) and (A4).

On the other hand, the perturbative treatment of the model when B is expanded as a power series in λ is far from transparent and does not appear to work, as may be discovered by making the attempt.

There is a whole family of models (of which the above is the simplest member) for which the TBA can be solved in similar fashion.

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